NONPERTURBATIVE APPROACH TO THE INFRARED PROBLEM IN MONOPOLE PROCESSES ☆

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The infrared problem of the quantum field theory of electric and magnetic charges is resolved by using functional methods, without recourse to perturbation theory.

1. Introduction

The infrared problem of the quantum field theory of electric and magnetic charges (QEMD) was solved in the realm of standard perturbation theory [1,2], and also by using coherent states [3]. The solution of the problem was achieved similarly as in QED, i.e. by taking into account the emission of soft real photons with energy below a certain threshold dictated by the experimental arrangements [4]. Due to the largeness of the magnetic coupling constant, the finite part of the infrared contribution yields superstrong radiation damping of the cross section in certain kinematic regions. The radiation damping factor is n-independent, and has important physical implications [2,5].

The treatment of the infrared problem by perturbative methods is of necessity formal, as QEMD is a strong coupling theory. In spite of this, the above results give us an important insight into the structure and physical content of the theory. This motivates us to study the infrared problem of QEMD by using functional methods, where all steps in the treatment can be carried out without recourse to perturbation theory [6].

2. A brief review of QEMD

Zwanziger succeeded in formulating an approach to QEMD based on the local lagrangian [7]. Let ψ_s be a set of spin-1/2 fields each carrying electric an magnetic charges e_s and g_s (dyons). The local lagrangian describing the electromagnetic interaction of these fields with each other is given by the expression

$$\mathscr{L}=\mathscr{L}_\gamma+\sum\bar{\psi}_s(\mathrm{i}\gamma\nabla_s-m_s)\psi_s\,, \tag{1}$$

where $\mathscr{L}_\gamma$ describes the electromagnetic field in terms of two potentials A^μ and B^μ,

$$\mathscr{L}_\gamma=\tfrac{1}{2}[n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)^*]-\tfrac{1}{2}[n\cdot(\partial\wedge B)]\cdot[n\cdot(\partial\wedge A)^*]+\tfrac{1}{2}[n\cdot(\partial\wedge A)]^2+\tfrac{1}{2}[n\cdot(\partial\wedge B)]^2\,, \tag{2}$$

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and $\nabla_s = \partial + i(e_s A + g_s B)$ is the covariant derivative. The lagrangian (1) is invariant under the $U(1)\times U(1)$ gauge transformations. Thus, in order to have a well-defined theory, it is necessary to fix the gauge. After adding a gauge-breaking term $\mathscr{L}_{GB}$ to $\mathscr{L}_\gamma$, and introducing a notation $A_a = (A, B)$ (which is explicitly covariant under the duality transformations), one can write the complete photon lagrangian in the form

$$\mathscr{L}'_\gamma \equiv \mathscr{L}_\gamma + \mathscr{L}_{GB} = \tfrac{1}{2} A^\mu_a K^{ab}_{\mu\nu} A^\nu_b , \tag{3}$$

where the kinetic operator K is invertible (its explicit form is determined by the choice of $\mathscr{L}_{GB}$). Zwanziger chose to work with the gauge-breaking term

$$\mathscr{L}_{GB} = -\tfrac{1}{2}[\partial(n\cdot A)]^2 - \tfrac{1}{2}[\partial(n\cdot B)]^2 , \tag{4a}$$

which leads to the free photon propagator

$$D^{ab}_{\mu\nu}(x) = -\{[\eta_{\mu\nu} - (\partial_\mu n_\nu + \partial_\nu n_\mu)(n\cdot\partial)^{-1}]\delta^{ab} + [\varepsilon_{\mu\nu\lambda\rho} n^\lambda \partial^\rho (n\cdot\partial)^{-1}]\epsilon^{ab}\}\Delta_F(x) \equiv R^{ab}_{\mu\nu}\Delta_F , \tag{4b}$$

where Δ_F satisfies the equation $\Box\Delta_F = -\delta$, and $K\cdot D = -\delta$.

3. Green's functions and S-matrix

Green's functions of QEMD can be studied by introducing the generating functional [2]

$$Z[J_e, J_g, \bar\eta, \eta] = N^{-1} \int \mathscr{D}A\, \mathscr{D}B\, \mathscr{D}\psi\, \mathscr{D}\bar\psi \exp\left(i \int (\mathscr{L} + \mathscr{L}_{GB} + J_e\cdot A + J_g\cdot B + \bar\eta\psi + \bar\psi\eta) \right) , \tag{5}$$

where J_e, J_g, $\bar\eta$ and η are external sources. The integration over $(A, B, \psi, \bar\psi)$ can be easily done when the interaction is absent $(e_s = g_s = 0)$, leading to

$$Z_0[J_a, \bar\eta, \eta] = \exp\left(\frac{i}{2} \int J^a D_{ab} J^b + i \int \bar\eta S \eta \right) . \tag{6}$$

where D_{ab} is the free photon propagator, eq. (4b), S is the free fermion propagator satisfying the equation $(i\gamma\cdot\partial - m)S = -1$, and $J_a = (J_e, J_g)$. The generating functional in the presence of the interaction can now be immediately written in the form

$$Z[J_a, \bar\eta, \eta] = \exp\left[-ie_a \int \frac{\delta}{\delta\eta} \left(\gamma^\mu \frac{1}{i} \frac{\delta}{\delta J^\mu_a} \right) \frac{\delta}{\delta\bar\eta} \right] Z_0[J_a, \bar\eta, \eta] . \tag{7}$$

where $e_a = (e_s, g_s)$. By using the formulae (3.61) and (3.69) of ref. [6] the above equation may be transformed into a more tractable form:

$$Z[J_a, \bar\eta, \eta] = \exp\left(\frac{i}{2} \int J^a D_{ab} J^b \right) \exp\left(-\frac{i}{2} \int \frac{\delta}{\delta\mathscr{A}^E_a} D_{ab} \frac{\delta}{\delta\mathscr{A}^E_b} \right) \exp\left(i \int \bar\eta G[\mathscr{A}^E]\eta \right) \exp(L[\mathscr{A}^E]) , \tag{8}$$

where $\mathscr{A}^E_a = \int D_{ab} J^b$ is the external electromagnetic potential, $G[\mathscr{A}^E]$ denotes the fermion propagator in the presence of the external potential,

$$[\gamma\cdot(i\partial - e_a \mathscr{A}^E_a) - m] G[\mathscr{A}^E] = -1 , \tag{9}$$

and $L[\mathscr{A}^E]$ describes the contribution of all closed fermion loops, $L[\mathscr{A}^E] = \mathrm{Tr}\ln(1 + e_a\gamma\cdot\mathscr{A}^E_a S)^{-1}$. Eq. (8) exhibits the important relation between the Green's functions of classical and quantum field theory.

All Green's functions may be obtained by functional differentiation of $Z[J_a, \bar\eta, \eta]$. The connection between the S-matrix and the Green's functions may be obtained by using the standard reduction formalism technique [6]. For the discussion of the infrared problem in the simplest case, we shall need *the S-matrix for the emission*

of n real photons when a dyon is scattered in a weak external field. The Green's function for the process is given by

$$G^{J}(x', x; z_1, ..., z_n) = \left(\prod_{i=1}^{n} \frac{\delta}{\mathrm{i}\delta j(z_i)} \right) \frac{\delta}{\delta\bar{\eta}(x')} \frac{\delta}{\delta\eta(x)} Z[J_a + j_a, \bar{\eta}, \eta] \Big|_{j,\bar{\eta},\eta=0} . \tag{10}$$

where j_a, $\bar{\eta}$ and η are the artifical (nonexternal) sources which are to be set equal to zero at the end of the calculation. The passage to the corresponding S-matrix in momentum space yields

$$S(p', p; k_1, ..., k_n) = \left(\frac{1}{\sqrt{n!}} \prod_{i=1}^{n} N_{\mathrm{B}}^{i} \epsilon_{\mu_i}^{a_i} \int \mathrm{d}z_i \exp(\mathrm{i}k_i z_i)\, K_{a_i b_i}^{\mu_i \nu_i} \frac{\delta}{\delta j_{b_i}^{\nu_i}(z_i)} \right)$$

$$\times \left(N_{\mathrm{F}} N_{\mathrm{F}}' \int \mathrm{d}x\, \mathrm{d}x' \exp[\mathrm{i}(p'x' - px)]\, \bar{u}(p') \vec{\mathcal{G}}_x \frac{\delta}{\delta\bar{\eta}(x')} \frac{\delta}{\delta\eta(x)} \overleftarrow{\mathcal{G}}_x u(p) \right) Z[J_a + j_a, \bar{\eta}, \eta] \Big|_0 . \tag{11}$$

Here, $N_{\mathrm{B}}^{i} = (2\pi)^{-3/2}(2\omega_i)^{-1/2}$ and $N_{\mathrm{F}} = (2\pi)^{-3/2} \cdot (m/E)^{1/2}$ are the boson and fermion state normalization factors, respectively, $\mathcal{G}_x = \mathrm{i}\gamma\cdot\partial - m$, $K_{ab}^{\mu\nu}$ is the photon kinetic operator, eq. (3), ϵ_{μ}^{a} is the photon polarization, and $u(p)$ is the free fermion state. The generating functional $Z[J_a + j_a, \bar{\eta}, \eta]$ is of the form (8), with $\mathcal{A}_a^{\mathrm{E}}$ replaced by $\mathcal{A}_a + \mathcal{A}_a^{\mathrm{E}}$, $\mathcal{A}_a = \int D_{ab} j^{b}$. By using the relations [6]

$$-K_{ab}^{\mu\nu} \frac{\delta}{\delta j_{b}^{\nu}} Z[j] \Big|_{j=0} = \frac{\delta}{\delta \mathcal{A}_{\mu}^{a}} Z[j = -K\mathcal{A}] \Big|_{j=0}, \quad \frac{\delta}{\delta\bar{\eta}(x')} \frac{\delta}{\delta\eta(x)} \exp\left(\mathrm{i} \int \bar{\eta} G[\mathcal{A} + \mathcal{A}^{E}] \eta \right) \Big|_{\bar{\eta},\eta=0} = G[\mathcal{A} + \mathcal{A}^{E}] , \tag{12}$$

the expression (11) for the S-matrix can be cast into the form

$$S(p', p; k_1, ..., k_n) = \left(\frac{(-1)^n}{\sqrt{n!}} \prod_{i=1}^{n} N_{\mathrm{B}}^{i} \epsilon_{\mu_i}^{a_i} \int \mathrm{d}z_i \exp(\mathrm{i}k_i z_i) \frac{\delta}{\delta \mathcal{A}_{a_i}^{\mu_i}(z_i)} \exp\left(-\frac{\mathrm{i}}{2} \int \frac{\delta}{\delta\mathcal{A}} D \frac{\delta}{\delta\mathcal{A}} \right) N_{\mathrm{F}} N_{\mathrm{F}}' \right.$$

$$\left. \times \int \mathrm{d}x\, \mathrm{d}x' \exp[\mathrm{i}(p'x' - px)]\, \bar{u}(p') \vec{D}_x G(x', x | \mathcal{A} + \mathcal{A}^{\mathrm{E}}) \overleftarrow{\mathcal{G}}_x u(p) \right|_{j=-K\mathcal{A}} . \tag{13}$$

This result is obtained by noting that those terms in eq. (11) which arise by differentiating $\exp[\frac{1}{2}\mathrm{i}\int(j+J)D(j+J)]$ over j should be dropped (as they do not contribute to the amplitude for emission of n real photons), and by omitting the constant term $\exp(\frac{1}{2}\mathrm{i}\int JDJ)$ from (13).

A physically important situation appears when real photons in the process (13) have vanishingly small momenta. The integration over soft photon momenta leads to logarithmic divergences, and we are faced with the so-called *infrared catastrophe*. In 1937, Bloch and Nordsieck [8] invented an approximation which provided a basis for the resolution of the infrared problem in QED. The problem is particularly suited to functional methods, where one can carry out all operations exactly (without using the perturbation expansion).

The physical content of the *Bloch–Nordsieck approximation* is the lack of fermion recoil [6]. The mathematical expression of this approximation is the replacement $\gamma^{\mu} \to p^{\mu}/m = v^{\mu}$ (where p^{μ} is an average fermion momentum) in eq. (9) for the classical fermion propagator $G[\mathcal{A}]$, which becomes

$$[v(\mathrm{i}\partial - e_a \mathcal{A}_a) - m]\, G_{\mathrm{BN}}(x, y | \mathcal{A}) = -\delta(x-y) . \tag{14}$$

This equation may be solved exactly for arbitrary $\mathcal{A}$, and the result is

$$G_{\mathrm{BN}}(x, y | \mathcal{A}) = \mathrm{i} \int_{0}^{\infty} \mathrm{d}\tau \exp(-\mathrm{i}m\tau)\, \delta(x - y - v\tau) \exp\left(-\mathrm{i}e_a \int_{0}^{\tau} \mathrm{d}\tau'\, v\cdot\mathcal{A}_a(x - v\tau') \right) . \tag{15}$$

In the dyon rest frame this expression reduces to the retarded propagator, and hence the closed loop functional $L[\mathscr{A}]$ vanishes in this approximation, $L_{\mathrm{BN}}[\mathscr{A}]=0$.

4. The infrared problem in potential scattering

Let us now consider the infrared problem in the simple case of the scattering of a dyon by a weak external field, but to all orders in the coupling constant $e_a=(e,g)$. To the first order in $\mathscr{A}^{\mathrm{E}}$, the classical electron propagator is given by

$$G(\mathscr{A}+\mathscr{A}^{\mathrm{E}})\simeq G(\mathscr{A})-G(\mathscr{A})e_a\hat{\mathscr{A}}_a F(\mathscr{A})\,, \tag{16}$$

where $\hat{\mathscr{A}}=\gamma\mathscr{A}$. For the bremsstrahlung process described by eq. (11) the term $G(\mathscr{A})$ vanishes because of momentum conservation, $p'\neq p+k_1+...+k_n$. After using (16) and the Bloch–Nordsieck approximation (15), the last line of the expression for the S-matrix (13) takes the form

$$\int \mathrm{d}x\,\mathrm{d}x'\,\exp[\mathrm{i}(p'x'-px)]\bar{u}(p')\vec{\mathscr{D}}_x\,G(x',x|\mathscr{A}+\mathscr{A}^{\mathrm{E}})\overleftarrow{\mathscr{D}}_x u(p)$$

$$\simeq\bar{u}(p')\int \mathrm{d}^4u[-e_a\hat{\mathscr{A}}^{\mathrm{E}}_a(u)]\exp[\mathrm{i}u(p'-p)]\exp\left(-\mathrm{i}\int \mathrm{d}z\,f^{\alpha}_{\nu}(z)\mathscr{A}^{\nu}_{a}(z)\right)u(p)\,, \tag{17}$$

where

$$f^{a}_{\nu}(z)=\int_{-\infty}^{0}\mathrm{d}\tau\,e^{a}v_{\nu}\delta[z-(u+v\tau)]+\int_{0}^{\infty}\mathrm{d}\tau\,e^{a}v'_{\nu}\delta[z-(u+v'\tau)] \tag{18}$$

is the classical dyon current. Now we use the relation [6]

$$\exp\left(-\frac{\mathrm{i}}{2}\int\frac{\delta}{\delta\mathscr{A}}D\frac{\delta}{\delta\mathscr{A}}\right)\exp\left(-\mathrm{i}\int f\mathscr{A}\right)=\exp\left(-\mathrm{i}\int f\mathscr{A}\right)\exp\left(\frac{\mathrm{i}}{2}\int fDf\right)$$

and obtain

$$S(p',p;k_1,...,k_n)=\int \mathrm{d}^4u\left(\frac{\mathrm{i}^n}{\sqrt{n}}\prod_{i=1}^{n}N'_{\mathrm{B}}\epsilon^{a_i}_{\mu_i}\int \mathrm{d}z\,\exp(\mathrm{i}k_iz_i)f^{\mu_i}_{a_i}(z_i)\right)$$

$$\times\{N_{\mathrm{F}}N'_{\mathrm{F}}\bar{u}(p')[-e_a\hat{\mathscr{A}}^{\mathrm{E}}_a(u)]\exp[\mathrm{i}u(p'-p)]u(p)\}\exp\left(\frac{\mathrm{i}}{2}\int fDf\right). \tag{19}$$

The last factor in the above expression defines the *virtual* soft photon contribution to the S-matrix,

$$S_{\mathrm{V}}\equiv\exp\left(\frac{\mathrm{i}}{2}\int fDf\right). \tag{20}$$

A direct calculation, by going over to momentum space and using the expressions (4b) and (18) for D and f, leads to

$$S_{\mathrm{V}}=\exp\left(\frac{1}{4\pi}B\right), \tag{21}$$

where

$$B = -\frac{\mathrm{i}}{8\pi^3}\int \frac{\mathrm{d}^4k}{k^2-\lambda^2}\,(e_a\bar{I}^\mu)R^{ab}_{\mu\nu}(e_b\bar{I}^\nu)\,,\quad \bar{I}_\mu \equiv \frac{p'_\mu}{p'\cdot k} - \frac{p_\mu}{p\cdot k}\,. \tag{22a}$$

R is the residue of D, eq. (4b), and λ is a small photon mass regulator. The gauge invariance of $\bar{I}^\mu(k_\mu\bar{I}^\mu=0)$ simplifies the above result,

$$B = \frac{\mathrm{i}}{8\pi^3}\int \frac{\mathrm{d}^4k}{k^2-\lambda^2}\,(e_a\bar{I}^\mu)\eta_{\mu\nu}(e_a\bar{I}^\nu)\,. \tag{22b}$$

This expression for B has a logarithmic *ultraviolet* divergence, which is to be removed by a certain regularization procedure. On the other hand, let us note that the present discussion describes only the infrared limit and not the approach to that limit. On the basis of this observation we can, following the treatments of refs. [4,9,10], replace $\bar{I}_\mu$ in eq. (22) by I_μ,

$$\bar{I}_\mu \to I_\mu = \frac{2p'_\mu + k_\mu}{2p'\cdot k + k^2} - \frac{2p_\mu + k_\mu}{2p\cdot k + k^2}\,. \tag{23a}$$

The expression for I_μ is gauge invariant, and has the same infrared limit as $\bar{I}_\mu$, but a different approach to that limit. The replacement $\bar{I}_\mu \to I_\mu$ changes the virtual soft photon contribution in such a way that it becomes finite in the ultraviolet regime,

$$B \to \frac{\mathrm{i}}{8\pi^3}\int \frac{\mathrm{d}^4k}{k^2-\lambda^2}\,(e_aI_\mu)\eta^{\mu\nu}(e_aI_\nu)\,, \tag{23b}$$

while the infrared limit is unchanged.

To evaluate the *real* photon contribution, we first note that the Fourier transform of $f^\mu_a(z)$ is proportional to $\bar{I}_\mu$,

$$\mathrm{i}\int \mathrm{d}z\,\exp(\mathrm{i}kz)f^\mu_a(z) = -\exp(\mathrm{i}ku)e_a\bar{I}^\mu(p',p)\,.$$

Then, the first line in the S-matrix (19) defines the real soft photon contribution:

$$S^{(n)}_{\mathrm{R}} = \frac{1}{\sqrt{n!}}\prod_{i=1}^{n} N'_{\mathrm{B}}\epsilon^{a_i}_{\mu_i}(-e_{a_i}\bar{I}^{\mu_i})\,. \tag{24}$$

Thus, we see that the infrared-divergent contributions of virtual and real photons to the S-matrix are *factorizable*,

$$S^{(n)}_{\mathrm{fi}} = S^{(0)}_{\mathrm{fi}}\cdot S\cdot S^{(n)}_{\mathrm{R}}\,. \tag{25}$$

Here, $S^{(0)}_{\mathrm{fi}}$ describes the elastic scattering amplitude.

To produce the probability we square S_{V} and S_{R} and sum over polarizations,

$$|S_{\mathrm{V}}|^2 = \exp\left(\frac{e_ae_a}{4\pi}\,2\,\mathrm{Re}\,B\right),\quad \sum|S^{(n)}_{\mathrm{R}}|^2 = \frac{1}{n!}\prod_{i=1}^{n}\prod_{j=1}^{n} N'_{\mathrm{B}}N'_{\mathrm{B}}(e_a\bar{I}^{\mu_i})R^{ab}_{\mu_i\nu_j}(e_b\bar{I}^{\nu_j})\,, \tag{26a,b}$$

where $R^{ab}_{\mu\nu} = \sum\epsilon^a_\mu\epsilon^{b*}_\nu$ is the residue of D, eq. (4b), $\bar{I}^{\mu_i} = \bar{I}^{\mu_i}(k=k_i)$. Taking into account the transversality of $\bar{I}^\mu$, and integrating over the phase space for n real photons with total energy below ΔE, one finds the probability

$$P_n(\Delta E) = |S^{(0)}_{\mathrm{fi}}|^2\,|S_{\mathrm{V}}|^2\,\frac{1}{n!}\prod_{i=1}^{n}\int_0^{\Delta E}\frac{\mathrm{d}^3k_i}{(2\pi)^3 2\omega_i}\,(e_a\bar{I}_{\mu_i})(-\eta^{\mu_i\nu_i})(e_a\bar{I}_{\nu_i})\,. \tag{27}$$

Summing over n produces the physical probability:

$$P(\Delta E) = |S^{(0)}_{\mathrm{fi}}|^2\exp\left(\frac{1}{4\pi}(2\,\mathrm{Re}\,B + \bar{B})\right). \tag{28}$$

recourse to perturbation theory. The results obtained here are n-independent, and confirm the earlier calculations based on formal perturbation theory [1,5] and the coherent state technique [3]. The prediction of super-strong radiation damping may be important for planning and analyzing experiments on monopole detection.

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